# Coulomb Drag in Quantum Hall Systems Near $\nu = 1/2$

by

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## Abstract

We use the composite fermion approach for theoretical studies of the Coulomb drag between two parallel layers of two-dimensional electron gases in the quantum Hall regime near Landau level filling fraction $\nu = 1/2$. Within the composite fermion approach, we use Boltzmann transport theory to determine the polarizability of the composite fermions. While this approach works at filling fraction $\nu = 1/2$, a straightforward expansion of the solution of the Boltzmann equation around $\nu = 1/2$ results in spurious divergences that stem from inaccuracies in the expansion at long wavelength. We then attempt to find expressions for the polarizability that are more accurate in this long wavelength limit. The excitation spectrum of the system in the absence of scattering consists of a discrete spectrum of $\delta$ function poles. We introduce tools to deal with such expressions, but we find that we cannot yield any exact results from this approach due to complications in determining the location of poles and the resulting residues.

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# Chapter 1

# Background Information

## 1.1 Introduction

Bilayer systems comprised of two dimensional electron gases (2DEG) at close proximity (separated by distances on the order of 100 Å) have been shown in various experiments to exhibit interesting phenomena, including Coulomb drag. In Coulomb drag experiments, two 2DEGs are arranged close together and interact via Coulomb forces. A current $I_2$ is driven in layer 2 from which momentum is transfered to layer 1 due to electron-electron interactions. The current in layer 1 is kept at zero by applying a voltage $V_1$. The ratio $\rho_D \equiv -V_1/I_2$ is called the Coulomb drag (or transresistivity) and can be written as [2, 3, 4]:

$$\rho_D = \frac{1}{2(2\pi)^2}\frac{h}{e^2}\frac{1}{Tn_1n_2}\int\frac{d\mathbf{q}}{(2\pi)^2}\int_0^\infty\frac{\hbar d\omega}{\sinh^2\frac{\hbar\omega}{2k_BT}}\, q^2|U_{sc}(\mathbf{q},\omega)|^2\mathrm{Im}\Pi_1(\mathbf{q},\omega)\mathrm{Im}\Pi_2(\mathbf{q},\omega) \tag{1.1}$$

where $T$ is the temperature, $n_i$ is the density in layer $i$, $U_{sc}$ is the screened inter-layer Coulomb interaction, and $\Pi_i(\mathbf{q},\omega)$ is the density-density response function in layer $i$. The Coulomb drag is due to scattering between the electrons in the two layers, which results from the screened inter-layer Coulomb interaction. The scattering events transfer momentum $\hbar q$ and energy $\hbar\omega$ between the layers with the restriction that $\hbar\omega < k_BT$, as enforced by the $\sinh^{-2}\frac{\hbar\omega}{2k_BT}$ term.

Using the composite fermion approach, we attempt to calculate $\rho_D$ between two

2DEGs in the quantum Hall regime (strong magnetic field and low temperature) at filling factor $\nu$ near $1/2$ by extending the results of Ussishkin-Stern [1]. The composite fermion picture was originally developed to explain the fractional quantum hall effect (FQHE), in which the Hall (transverse) resistance of a current-carrying 2DEG subject to a uniform magnetic field (applied perpendicular to the sample) has plateaus where its value is quantized at $R_H = 2\pi\hbar/\nu e^2$.

## 1.2 The Quantum Hall Effect

To understand the FQHE, consider the Hamiltonian for a system of interacting electrons:

$$H = \sum_j \frac{1}{2m_b}\left[\frac{\hbar}{i}\nabla_j + \frac{e}{c}\mathbf{A}(\mathbf{r}_j)\right]^2 + \frac{e^2}{\epsilon}\sum_{j<k}\frac{1}{|\mathbf{r}_j - \mathbf{r}_k|} + \sum_j U(\mathbf{r}_j) + g\mu\mathbf{B}\cdot\mathbf{S} \qquad (1.2)$$

where $m_b$ is the band mass of the electron, $-e$ is the charge of the electron, and $c$ is the speed of light. The last term is the Zeeman energy which we can neglect in the GaAs heterostructures we are considering. We will assume spinless (or rather spin-polarized) electrons for the rest of this discussion. $U$ is the disorder potential, but we will assume there is no disorder. The first term is the integer quantum hall effect (IQHE) Hamiltonian: the kinetic energy of noninteracting electrons in the presence of a constant external magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. The single-electron Hamiltonian $H = \frac{1}{2m_b}(\mathbf{p} + \frac{e}{c}\mathbf{A})^2$ gives rise to quantized energy levels called the Landau levels:

$$E_n = (n + \frac{1}{2})\hbar\omega_c, \qquad (1.3)$$

where $n$ is the Landau level index and

$$\omega_c = \frac{eB}{m_b c}. \qquad (1.4)$$

Each Landau level has a degeneracy of $B/\phi_0$ states per unit area where the flux quantum $\phi_0$ is defined as $2\pi\hbar c/e$. The number of filled Landau levels, called the filling fraction, is the electron density $n_e$ divided by the degeneracy:

$$\nu = n_e 2\pi\hbar c/eB = n_e\phi_0/B \tag{1.5}$$

The IQHE occurs when the filling factor $\nu$ is an integer, i.e. the lowest $\nu$ Landau levels are filled, and at sufficiently low temperatures and high magnetic fields, the energy gap $\hbar\omega_c \gg k_B T$. Hence, the electrons in the occupied Landau levels cannot move to the higher unoccupied Landau levels by absorbing a thermal phonon and thus $R_H$ remains constant and the longitudinal resistance falls to zero. However, such a simple explanation does not suffice to explain the FQHE, which occurs at certain fractional values of $\nu$.

## 1.3 The Composite Fermion Approach

At the very high magnetic fields at which the FQHE is observed, the level spacing is so large that all the electrons are confined to the lowest Landau level, so the kinetic energy is a constant that we can neglect. So the problem of describing the FQHE is reduced to solving the following Hamiltonian in the lowest Landau level:

$$H' = \frac{e^2}{\epsilon}\sum_{j<k}\frac{1}{|\mathbf{r}_j - \mathbf{r}_k|} \tag{1.6}$$

This Hamiltonian illustrates the strongly correlated nature of the FQHE system. Despite the apparent simplicity of the Hamiltonian, the systematic solution of the ground state of a FQHE system of $10^{11}$ electrons per square centimeter is an intractable problem. Nonetheless, we can make progress on the problem by employing Chern-Simons theory. Suppose $\psi(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ is an eigenfunction of $H$, with $\mathbf{r}_j$ the position of the $j^{th}$ electron. Let us consider the transformed wavefunction

$$\psi_{CS}(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \left[\prod_{j<k} e^{-i\tilde{\phi}\theta(\mathbf{r}_j - \mathbf{r}_k)}\right]\psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) \tag{1.7}$$

where $\tilde{\phi} = 2m$ and $\theta(\mathbf{r}_j - \mathbf{r}_k)$ is the angle formed by the vector $\mathbf{r}_j - \mathbf{r}_k$ with the $x$-axis. Note that if $\psi$ obeys Fermi statistics, then $\psi_{CS}$ does as well since $\tilde{\phi}$ is an even integer. If $\psi$ is a solution of the Schrodinger equation $H\psi = E\psi$, then $\psi_{CS}$ is an eigenfunction of the Chern-Simons Hamiltonian:

$$H_{CS} = \sum_j \frac{1}{2m^*}\left[\frac{\hbar}{i}\nabla_j + \frac{e}{c}\mathbf{A}(\mathbf{r}_j) - \frac{e}{c}\mathbf{A}_{CS}(\mathbf{r}_j)\right]^2 + \frac{e^2}{\epsilon}\sum_{j<k}\frac{1}{|\mathbf{r}_j - \mathbf{r}_k|}{}^1, \tag{1.8}$$

with the Chern-Simons vector potential

$$\mathbf{A}_{CS}(\mathbf{r}_j) = i\nabla_j\left[\prod_k e^{-i\tilde{\phi}\theta(\mathbf{r}_j - \mathbf{r}_k)}\right] = \frac{\tilde{\phi}\phi_0}{2\pi}\sum_{k=1}^{N}\frac{\hat{\mathbf{z}} \times (\mathbf{r}_j - \mathbf{r}_k)}{|\mathbf{r}_j - \mathbf{r}_k|^2}, \tag{1.9}$$

The corresponding Chern-Simons magnetic field $\mathbf{B}_{CS}$ is then given by

$$\mathbf{B}_{CS} = \nabla \times \mathbf{A}_{CS}(\mathbf{r}) = \tilde{\phi}\phi_0\delta(\mathbf{r} - \mathbf{r}_i) = \tilde{\phi}\phi_0 n(\mathbf{r}) \tag{1.10}$$

where $n(\mathbf{r}) = \sum_i \delta(\mathbf{r} - \mathbf{r}_i)$ is the electron density.

Intuitively, the Chern-Simons transformation can be seen as attaching to each electron an infinitely thin, massless solenoid carrying $\tilde{\phi} = 2m$ flux quanta antiparallel to $\mathbf{B}$, thereby transforming it into a composite fermion. Please note that the Chern-Simons field is a fictitious, unobservable field introduced in order to simplify the problem at hand. The next step is to make a mean field approximation to $H_{CS}$, in which we assume the electron density is uniform and replace the position-dependent Chern-Simons field $\mathbf{B}_{CS}$ by its average value $\langle B_{CS}\rangle = n_e\tilde{\phi}\phi_0$. Hence, the composite fermions feel a reduced mean field:

$$\Delta B = B - \langle B_{CS}\rangle = B - n_e\tilde{\phi}\phi_0 = B - 2mn_e\phi_0 \tag{1.11}$$

[1] $m^*$ is an effective mass. At the mean field level, we have $m^* = m_b$, which is not correct because we expect that the effective mass should be renormalized by interactions. We estimate the value of the effective mass following [5]. Assuming that the electron interaction energy is much less than the spacing between Landau levels, we can neglect Landau level mixing so all energies of interaction must then be proportional to the electron-electron interaction energy scale $e^2(4\pi n_e)^{1/2}/\epsilon$. From dimensional analysis, we see that the effective mass should have the form $m^* = \frac{\hbar^2(4\pi n_e)^{1/2}\epsilon}{e^2 C}$, where $C$ is a dimensionless constant estimated as 0.3.[5] With $\epsilon = 12.6$ (for GaAs), a field of $B = 10T$ and a filling fraction of $\nu = \frac{1}{2}$, we can estimate $m^* \approx 4m_b$.

Note that $\Delta B = 0$ at filling fraction $\nu = \frac{n_e \phi_0}{B} = \frac{1}{2m}$. Thus, at $\nu = \frac{1}{2}$, the system can be identified as quasiparticles in zero magnetic field (within the mean field approximation). Thus, the system is a Fermi liquid rather than a quantum Hall state. At filling fraction slightly away from 1/2, the applied magnetic field and the Chern-Simons flux do not exactly cancel, so the composite fermions experience a nonzero effective magnetic field:

$$\Delta B = B - \langle B_{CS} \rangle = B - 2n_e \phi_0 \tag{1.12}$$

Then we have for the mean field Hamiltonian:

$$H_0 = \frac{1}{2m^*} \int d\mathbf{r} \psi^\dagger_{CS}(\mathbf{r}) \left[ \frac{\hbar}{i} \nabla + \frac{e}{c} \Delta \mathbf{A}(\mathbf{r}) \right]^2 \psi_{CS}(\mathbf{r}) \tag{1.13}$$

where $\Delta\mathbf{A}$ is a mean field vector potential which satisfies $\nabla \times (\Delta\mathbf{A}) = \Delta\mathbf{B}$. This mean-field Hamiltonian describes fermions in a magnetic field $\Delta\mathbf{B}$, so the energy levels are simply the Landau levels as in the IQHE, but here they are the energy levels for the Chern-Simons wavefunctions of the composite fermions. Recall that the IQHE occurs at filling fractions $\nu = n\phi_0/B$, so for $p \equiv n\phi_0/\Delta B$, we have an IQHE of composite fermions. For arbitrary $m$, we have for the filling fraction:

$$\nu = \frac{n\phi_0}{B} = \frac{n\phi_0}{B_{1/2m} + \Delta B} = \frac{1}{2m + \frac{1}{p}} = \frac{p}{2mp + 1} \tag{1.14}$$

Most of the filling fractions observed in experiments are given by the form above.

## 1.4 The Random Phase Approximation

To make further progress on calculating physical quantities that characterize a composite fermion system, such as the Hall conductivity, we will employ the random phase approximation (RPA), which moves beyond the mean field approximation and takes into account interactions between composite fermions, such as the change in the Chern-Simons vector potential that results from the movement of composite fermions and thus the flux quanta they carry. We start with a heuristic derivation of the RPA: Consider the Chern-Simons magnetic field $\delta B_{CS} = \tilde{\phi}\phi_0\, \delta n$ due to an excess density $\delta n$ carrying Chern-Simons flux of $\tilde{\phi}\, \delta n$. We deduce from Maxwell's equations that $\mathbf{E}_{CS} = \frac{\mathbf{v}}{c} \times \mathbf{B}_{CS}$, giving:

$$\delta E_{CS} = \frac{\mathbf{v}}{c} \times \tilde{\phi}\phi_0\, \delta n \hat{\mathbf{z}} \tag{1.15}$$

Noting that $\mathbf{j} = -n_e e \mathbf{v}$, we have for the Chern-Simons electric field:

$$\mathbf{E}_{CS} = -\frac{\mathbf{j}}{n_e e c} \times \tilde{\phi}\frac{2\pi\hbar c}{e} n_e \hat{\mathbf{z}} = \frac{2\pi\hbar\tilde{\phi}}{e^2}(\hat{\mathbf{z}} \times \mathbf{j}) \tag{1.16}$$

We then define the composite fermion resistivity tensor as follows [6]:

$$\mathbf{E}_{CS} = -\rho_{CS}\mathbf{j} \tag{1.17}$$

where

$$\rho_{CS} = \frac{2\pi\hbar\tilde{\phi}}{e^2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{1.18}$$

The resistivity matrix, defined such that $\mathbf{E} = \rho\mathbf{j}$, is given by

$$\rho = \rho_{CF} + \rho_{CS} \tag{1.19}$$

where $\rho_{CF} = \sigma_{CF}^{-1}$, and $\sigma_{CF}$ is defined by

$$\mathbf{j} = \sigma_{CF}(\mathbf{E} + \mathbf{E}_{CS}(\mathbf{j})) \tag{1.20}$$

$\sigma_{CF}$ is the composite fermion conductivity that characterizes the response of the composite fermions to both the physical electric field $\mathbf{E}$ and the self-consistently induced Chern-Simons electric field $\mathbf{E}_{CF}$. This is the essential element of the RPA: We treat the composite fermions as free fermions subject to the total electric field, i.e. the physical field and the self-consistently induced Chern-Simons field.

We now present a more formal derivation of the RPA. [5] Consider the linear response function $K_{\mu\nu}(\mathbf{q},\omega)$ defined as follows:

$$j_\mu(\mathbf{q},\omega) = \frac{e}{c} K_{\mu\nu}(\mathbf{q},\omega) A_\nu^{\text{ext}}(\mathbf{q},\omega) \tag{1.21}$$

where $A_\nu^{\text{ext}}$ is an external perturbing potential ($A_0$ is the scalar electrostatic potential, and $A_x$ and $A_y$ are the components of the magnetic vector potential) with frequency $\omega$ and wavevector $\mathbf{q}$. $j_\mu$ is the induced change in the particle density ($\mu = 0$) or current ($\mu = x, y$). Following [5], we choose $\mathbf{q} \,||\, \hat{\mathbf{x}}$, and we work in the Coulomb gauge so that $\mathbf{A} = A_1 \hat{\mathbf{y}}$, and the longitudinal part of $\mathbf{A}$ is zero (i.e. $A_x = 0$). Thus, the longitudinal part of $\mathbf{j}$ is simply $(\omega/q) j_0$. Therefore, we can regard $K_{\mu\nu}$ as a $2 \times 2$ matrix in which the indices take on the values 0 and 1 where the index 0 denotes the time component and 1 the transverse or $\hat{\mathbf{y}}$-direction.

We now consider the effect of the Coulomb interaction. We define the response to the field internally induced by the Coulomb interaction as follows:

$$j_\mu(\mathbf{q},\omega) = -\frac{e}{c} V_{\mu\nu}^{-1}(\mathbf{q},\omega) A_\nu^{\text{v-ind}}(\mathbf{q},\omega) \tag{1.22}$$

where

$$V = \begin{bmatrix} v(q) & 0 \\ 0 & 0 \end{bmatrix} \tag{1.23}$$

and

$$v(q) = \frac{2\pi e^2}{\epsilon q} \tag{1.24}$$

is the Fourier transform of the Coulomb potential

$$v(\mathbf{r}_j - \mathbf{r}_k) = \frac{e^2}{\epsilon} \sum_{j<k} \frac{1}{|\mathbf{r}_j - \mathbf{r}_k|}. \tag{1.25}$$

We define the total physical vector potential that includes both the external potential and that induced by the Coulomb interaction:

$$A_\nu^{\text{total-physical}} = A_\nu^{\text{ext}} + A_\nu^{\text{v-ind}} \tag{1.26}$$

The polarizability $\Pi$, which gives the density and current response to the total physical field (i.e. what is measured experimentally) is defined as follows:

$$j_\mu(\mathbf{q}, \omega) = \frac{e}{c}\Pi_{\mu\nu}(\mathbf{q}, \omega) A_\nu^{\text{total-physical}}(\mathbf{q}, \omega) \tag{1.27}$$

Combining equations 1.21, 1.22, 1.26, and 1.27, writing $A^{\text{total-physical}}$ as $\Pi^{-1}j$, and so on, we have:

$$K^{-1} = \Pi^{-1} + V \tag{1.28}$$

We now define the Chern-Simons interaction matrix $C$ such that

$$j_\mu(\mathbf{q}, \omega) = -\frac{e}{c} C_{\mu\nu}(\mathbf{q}, \omega) A_\nu^{\text{CS-ind}}(\mathbf{q}, \omega) \tag{1.29}$$

where $A_\nu^{\text{CS-ind}}$ is the induced Chern-Simons vector potential. In order to find an expression for $C$, it is convenient to introduce a conversion matrix [6]:

$$T = e \begin{bmatrix} \frac{i\sqrt{i\omega}}{q} & 0 \\ 0 & \frac{1}{\sqrt{i\omega}} \end{bmatrix} \tag{1.30}$$

in order to convert between the vector potential $A = (A_0, A_y)$ and the electric field $\mathbf{E} = \nabla A_0 - \partial \mathbf{A}/\partial t = (E_x, E_y) = (-iqA_0, -i\omega A_1)$, as well as the tensor $j = (j_0, j_y)$ and $\mathbf{j} = (j_x, j_y)$. Hence

$$\mathbf{E} = -i\sqrt{-i\omega}\, T^{-1} A \tag{1.31}$$

$$\mathbf{j} = -i\sqrt{-i\omega}\, T j \tag{1.32}$$

Recalling that $\mathbf{j} = -\rho_{CS}^{-1}\mathbf{E}_{CS}$, we have for $C$:

$$C = T^{-1}\rho_{CS}^{-1}T^{-1} = \frac{1}{2\pi\hbar\tilde{\phi}} \begin{bmatrix} 0 & iq \\ -iq & 0 \end{bmatrix} \tag{1.33}$$

Note that this conversion also allows us to rewrite equation 1.27 as:

$$\mathbf{j} = \sigma \mathbf{E}^{\text{total-physical}} \tag{1.34}$$

which in turn implies that:

$$\sigma_{xx} = \frac{\omega e^2}{iq^2}\Pi_{00} \tag{1.35}$$

$$\sigma_{yy} = \frac{e^2}{i\omega}\Pi_{11} \tag{1.36}$$

Consider the response function $\tilde{K}$ that gives the current and density response to the total field $A_\nu^{\text{total}} = A_\nu^{\text{ext}} + A_\nu^{\text{CS-ind}} + A_\nu^{\text{v-ind}}$:

$$j_\mu(\mathbf{q},\omega) = e\tilde{K}_{\mu\nu}(\mathbf{q},\omega)A_\nu^{\text{total}}(\mathbf{q},\omega) \tag{1.37}$$

Combining equations 1.21, 1.22, 1.29, and 1.37, we have:

$$K^{-1} = \tilde{K}^{-1} + C^{-1} + V \tag{1.38}$$

We then find:

$$\Pi^{-1} = \tilde{K}^{-1} + C^{-1} \tag{1.39}$$

The RPA consists of approximating $\tilde{K}^{-1}$ by $K^0$, the response of non-interacting mean field composite fermions, for which we will later derive an expression. So, finally we obtain [5]:

$$\Pi^{-1} = (K^0)^{-1} + C^{-1} \tag{1.40}$$

It follows that the density-density response function for a given layer $i$ is:

$$\Pi_{00(i)}(\mathbf{q},\omega) = \frac{K^0_{00(i)}(\mathbf{q},\omega)}{1 - \frac{8i\pi\hbar}{q}K^0_{01(i)}(\mathbf{q},\omega) - \left(\frac{4\pi\hbar}{q}\right)^2\Delta_{(i)}(\mathbf{q},\omega)}. \tag{1.41}$$

where

$$\Delta_{(i)}(\mathbf{q},\omega) = K^0_{00(i)}(\mathbf{q},\omega)K^0_{11(i)}(\mathbf{q},\omega) + \left(K^0_{01(i)}(\mathbf{q},\omega)\right)^2. \tag{1.42}$$

The RPA response functions $K^0_{\mu\nu}$ are related to components of the CF conductivity tensor $\tilde{\sigma}$ as follows [5]:

$$\frac{1}{\tilde{\sigma}^{(i)}_{xx}(\mathbf{q},\omega)} = \frac{iq^2}{\omega e^2}\left[\frac{1}{K^0_{00(i)}(\mathbf{q},\omega)} - \frac{1}{K^0_{00(i)}(\mathbf{q},0)}\right];$$

$$\tilde{\sigma}^{(i)}_{yy}(\mathbf{q},\omega) = -\frac{ie^2}{\omega}\left[K^0_{11(i)}(\mathbf{q},\omega) - K^0_{11(i)}(\mathbf{q},0)\right];$$

$$\tilde{\sigma}^{(i)}_{xy} = -\tilde{\sigma}^{(i)}_{yx} = \frac{ie^2}{q}K^0_{01(i)}(\mathbf{q},\omega). \tag{1.43}$$

We set $[K^0_{00(i)}(\mathbf{q},0)]^{-1} = 0$ and the Landau diamagnetism $K^0_{11(i)}(\mathbf{q},0) = -\frac{q^2}{24\pi m^*}$.[1] We add the diamagnetic term by hand because the Boltzmann equation, which we will use to calculate the conductivity, does not correctly describe the Landau diamagnetic contribution to the transverse static response. We then determine the response functions $K^0_{\mu\nu}(\mathbf{q},\omega)$ from the conductivities, allowing us to arrive at an expression for $\Pi_{00}$.

## 1.5 Calculating the conductivities from the Boltzmann equation

We can calculate the components of the conductivity tensor by solving the Boltzmann transport equation for the CF distribution function as follows. Consider the distribution function $f(\mathbf{r}, \mathbf{p}, t)$ in phase space for quasiparticles at position $\mathbf{r}$ with momentum $\mathbf{p}$. We determine the time rate of change of this distribution function by means of the Boltzmann equation, which essentially is the statement of conservation of particle number in phase space:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{\partial \mathbf{p}}{\partial t}\cdot\nabla_{\mathbf{p}} f = \left(\frac{df}{dt}\right)_{collisions} \tag{1.44}$$

The right hand side is the collision term which we will estimate using the relaxation time approximation, i.e. we will assume that the distribution of quasiparticles emerging from collisions in an interval $dt$ is $dt/\tau$ multiplied by the equilibrium distribution function $f^\circ$, where $1/\tau$ is the probability per unit time that a quasiparticle composite fermion experiences a collision. Recall that at $B_{1/2}$ the composite fermions experience zero effective magnetic field, so we regard the distribution function at $B_{1/2}$ as the equilibrium distribution function, and $\Delta B = B - B_{1/2}$ the [uniform] magnetic field applied to the system. A weak electric field $\mathbf{E}$ at wavevector $\mathbf{q}\,||\hat{\mathbf{x}}$ and frequency $\omega$ is applied such that the response of system is linear and all perturbations are proportional to $e^{i(qx-\omega t)}$, i.e. $\delta f(\mathbf{p},\mathbf{r},t) = \delta f(\mathbf{p},\mathbf{q},\omega)e^{i(qx-\omega t)}$. So we have:

$$\partial f/\partial t + [\mathbf{v}\cdot\nabla + e(\mathbf{E} + \mathbf{v}\times\Delta\mathbf{B}/c)\cdot\nabla_{\mathbf{p}}]\,f = -[f - f^\circ(\mathbf{p})]/\tau, \tag{1.45}$$

where $\mathbf{v} \equiv \mathbf{p}/m^*$.

To solve the Boltzmann equation, we take

$$f(\mathbf{p},\mathbf{r},t) = f^\circ(\mathbf{p}) + \delta f(\mathbf{p},\mathbf{r},t) = f^\circ(\mathbf{p}) + \delta f(\mathbf{p},\mathbf{q},\omega)e^{i(qx-\omega t)}, \tag{1.46}$$

where $\delta f$ is first order in the external fields $\mathbf{E}$ and $\Delta\mathbf{B}$. In accordance with the uncertainty principle, there exists an uncertainty $\hbar\mathbf{q}$ in the momentum $\mathbf{p}$, and $\hbar\omega$ in the energy $E_p$. We can ignore the uncertainty principle if we are in a semiclassical regime in which the energy levels are closely spaced relative to the other energy scales in the system, i.e. the fluctuations of the Fermi surface are small relative to the characteristic width of the Fermi surface, which is $k_BT$ for the energy, $k_BT/v_F$ for the momentum. This translates into low effective magnetic field and long wavelength: $\hbar\omega \ll k_BT$ and $\hbar q v_F \ll k_BT$. In the semiclassical regime, we can regard the quasiparticles as localized wave packets subject to an effective magnetic field $\Delta B$.

The linearized, Fourier transformed Boltzmann equation is then:

$$i(\mathbf{q}\cdot\mathbf{v} - \omega - i/\tau)\delta f + \frac{e}{c}(\mathbf{v}\times\Delta\mathbf{B})\cdot\nabla_{\mathbf{p}}(\delta f) = -e\mathbf{E}\cdot\nabla_{\mathbf{p}}f^\circ - \frac{e}{c}(\mathbf{v}\times\Delta\mathbf{B})\cdot\nabla_{\mathbf{p}}f^\circ \tag{1.47}$$

We choose $\Delta\mathbf{B}$ along the $z$ axis with $\mathbf{q}$ and $\mathbf{v}$ restricted to the $xy$ plane in this two-dimensional electron gas. This gives us:

$$-\frac{e}{c}(\mathbf{v}\times\Delta\mathbf{B})\cdot\nabla_{\mathbf{p}}(\delta f)=\frac{e\Delta Bv}{c}\frac{1}{m^*}\nabla_{\mathbf{v}}(\delta f)=\omega_c^*\frac{\partial(\delta f)}{\partial\theta}, \tag{1.48}$$

where $\theta$ is the angle $\mathbf{p}\equiv m\mathbf{v}$ makes with the $x$ axis, and $\omega_c^*=\frac{e\Delta B}{m^*c}$.

Since $f^\circ$ is the equilibrium distribution, it is only a function of the energy $E_p^\circ$, so we have:

$$\nabla_{\mathbf{p}}f^\circ=\nabla_{\mathbf{p}}E_p\frac{\partial f^\circ}{\partial E_p^\circ}=\mathbf{v}\frac{\partial f^\circ}{\partial E_p^\circ} \tag{1.49}$$

Furthermore, for a degenerate Fermi liquid at zero temperature, the equilibrium distribution is the Fermi-Dirac distribution at zero temperature (i.e. a step function), so we have:

$$\frac{\partial f^\circ}{\partial E_p^\circ}=-\delta(E_p^\circ-E_F) \tag{1.50}$$

where $E_F$ is the Fermi energy and $E_p^\circ=p^2/2m^*$. Note that $\delta(E_p^\circ-E_F)$ restricts $\mathbf{p}$ to the Fermi surface. So we have for the last term in equation 1.47:

$$\frac{e}{c}(\mathbf{v}\times\Delta\mathbf{B})\cdot\nabla_{\mathbf{p}}f^\circ=\frac{e}{c}(\mathbf{v}\times\Delta\mathbf{B})\cdot\mathbf{v}\delta(E_p^\circ-E_F)=0 \tag{1.51}$$

The Boltzmann equation then simplifies to:

$$\left(-i(\omega+i/\tau-qv_F\cos\theta)+\omega_c^*\frac{\partial}{\partial\theta}\right)\delta f=-ev_F\mathbf{E}\cdot\hat{\mathbf{n}}(\theta) \tag{1.52}$$

where $\hat{\mathbf{n}}(\theta)=(\cos\theta,\sin\theta)$.

We have the relations

$$\mathbf{j}(\mathbf{q},\omega)=\sigma(\mathbf{q},\omega)\mathbf{E}(\mathbf{q},\omega) \tag{1.53}$$

and

$$\mathbf{j}(\mathbf{q},\omega)=\frac{e}{2m\pi^2}\int d\mathbf{p}\ \mathbf{p}\delta f(\mathbf{p},\mathbf{q},\omega) \tag{1.54}$$

$\delta(E_p^\circ-E_F)$ restricts $\mathbf{p}$ to the Fermi surface, enforcing that we integrate over the Fermi surface such that:

$$\mathbf{j} = -\frac{ep_F^2}{2m\pi^2}\int d\theta \hat{\mathbf{n}}(\theta)\delta f(\theta) \tag{1.55}$$

Once we solve equation 1.52 for $\delta f$, we can find the conductivities from equations 1.53 and 1.55 [7]:

$$\sigma_{xx} = i\pi N \sum_{n=0}^{\infty} \frac{(\omega + i/\tau)\omega_c^*}{(kv_F)^2} \frac{J_n^2(kv_F/\omega_c^*)}{((\omega + i/\tau)/\omega_c^*)^2 - n^2)} \tag{1.56}$$

$$\sigma_{yy} = i\pi N \sum_{n=0}^{\infty} \frac{(\omega + i/\tau)[J_n'(kv_F/\omega_c^*)]^2}{\omega_c^*(1+\delta_{n0})\left((\omega + i/\tau)/\omega_c^*)^2 - n^2\right)} \tag{1.57}$$

$$\sigma_{zz} = \pi N \sum_{n=0}^{\infty} \frac{(\omega + i/\tau)^2}{kv_F\omega_c^*} \frac{J_n(kv_F/\omega_c^*)J_n'(kv_F/\omega_c^*)}{(1+\delta_{n0})\left((\omega + i/\tau)/\omega_c^*)^2 - n^2\right)} \tag{1.58}$$

where $N \equiv 3n_e e^2/m^*\omega_c^*$. However, these expressions are not easily analytically integrated, so a more tractable form for the conductivities was approximated by ignoring scattering (i.e. setting $\tau \to \infty$) and expanding equation 1.52 in powers of $\Delta B$, or equivalently powers of $\omega_c^*$, by expanding the distribution function as follows:

$$\delta f = f_0 + f_1 + f_2 + \ldots \tag{1.59}$$

where $f_2$ is second-order in $\Delta B$. We can then solve equation 1.52 iteratively:

$$f_0(\theta) = \frac{-iev_F\mathbf{E}\cdot\hat{\mathbf{n}}(\theta)}{\omega - qv_F\cos\theta} \tag{1.60}$$

$$f_1(\theta) = \frac{i\omega_c^*\frac{d}{d\theta}f_0(\theta)}{\omega - qv_F\cos\theta} \tag{1.61}$$

$$f_2(\theta) = \frac{i\omega_c^*\frac{d}{d\theta}f_1(\theta)}{\omega - qv_F\cos\theta} \tag{1.62}$$

we thus have

$$
\begin{aligned}
f_0(\theta) &= -ev_F i\frac{[E_x\cos\theta + E_y\sin\theta]}{(\omega - qv_{\mathrm{F}}\cos\theta)} && (1.63)\\
f_1(\theta) &= -ev_F\omega_c^*\frac{[E_x\omega\sin\theta + E_y(qv_{\mathrm{F}} - \omega\cos\theta)]}{(\omega - qv_{\mathrm{F}}\cos\theta)^3} && (1.64)\\
f_2(\theta) &= -ev_F i\omega_c^{*2}\frac{[E_x\omega(\omega\cos\theta + qv_{\mathrm{F}}(\cos 2\theta - 2))]}{(\omega - qv_{\mathrm{F}}\cos\theta)^5} && (1.65)\\
&+ \frac{[E_y\sin\theta(\omega^2 - 3q^2v_{\mathrm{F}}^2 + 2\omega qv_{\mathrm{F}}\cos\theta)]}{(\omega - qv_{\mathrm{F}}\cos\theta)^5} && (1.66)
\end{aligned}
$$

Expanding in powers of $\omega_c^*$, we have

$$
\begin{aligned}
\sigma_{xx} &= \sigma_{xx}^{(0)} + \sigma_{xx}^{(2)} + \ldots && (1.67)\\
\sigma_{yy} &= \sigma_{yy}^{(0)} + \sigma_{yy}^{(2)} + \ldots && (1.68)\\
\sigma_{xy} &= \sigma_{xy}^{(1)} + \sigma_{xy}^{(3)} + \ldots && (1.69)
\end{aligned}
$$

We then integrate the $f_i$ using equation 1.55 to obtain the conductivities, with $\delta \equiv \frac{\omega}{qv_{\mathrm{F}}}$ and $R \equiv v_F/\omega_c^*$, where $R$ is the radius of the cyclotron orbit exhibited by the quasiparticles under the influence of $\Delta B$:

$$
\begin{aligned}
\sigma_{xx}^{(0)} &= \left[\frac{-ep_{\mathrm{F}}^2}{(2\pi\hbar)^2 m}\right]\int d\theta\cos\theta\frac{-ei\cos\theta}{(\omega - qv_{\mathrm{F}}\cos\theta)} = \frac{ik_{\mathrm{F}}e^2}{q2\pi\hbar}\delta\left[\frac{1}{\sqrt{1-\frac{1}{\delta^2}}} - 1\right] && (1.70)\\
\sigma_{yy}^{(0)} &= \left[\frac{-ep_{\mathrm{F}}^2}{(2\pi\hbar)^2 m}\right]\int d\theta\sin\theta\frac{-ei\sin\theta}{(\omega - qv_{\mathrm{F}}\cos\theta)} = \frac{ik_{\mathrm{F}}e^2}{q2\pi\hbar}\delta\left[1 - \sqrt{1-\frac{1}{\delta^2}}\right]. && (1.71)
\end{aligned}
$$

$$
\sigma_{yx}^{(1)} = \left[\frac{-ep_{\mathrm{F}}^2}{(2\pi\hbar)^2 m}\right]\int d\theta\sin\theta\frac{-e\omega_c^*\omega\sin\theta}{(\omega - qv_{\mathrm{F}}\cos\theta)^3} = \frac{e^2}{4\pi\hbar}\frac{k_{\mathrm{F}}}{q}\frac{1}{qR}\frac{\delta}{(\delta^2-1)^{3/2}} \qquad (1.72)
$$

$$\sigma^{(2)}_{xx} = \left[\frac{-ep_{\rm F}^2}{(2\pi\hbar)^2 m}\right] \int d\theta \cos\theta \frac{-ei\omega_c^{*2}[\omega(\omega\cos\theta + qv_{\rm F}(\cos 2\theta - 2))]}{(\omega - qv_{\rm F}\cos\theta)^5} \tag{1.73}$$

$$= \frac{ik_{\rm F}e^2}{q2\pi\hbar}\delta\left[\frac{i\delta}{\sqrt{(1-\delta^2)^5}} \times \frac{1}{2(qR)^2}\left(1 - \frac{5}{4}\frac{1}{1-\delta^2}\right)\right] \tag{1.74}$$

$$\sigma^{(2)}_{yy} = \left[\frac{-ep_{\rm F}^2}{(2\pi\hbar)^2 m}\right] \int d\theta \sin\theta \frac{-ei\omega_c^{*2}\sin\theta(\omega^2 - 3q^2v_{\rm F}^2 + 2\omega qv_{\rm F}\cos\theta)}{(\omega - qv_{\rm F}\cos\theta)^5} \tag{1.75}$$

$$= \frac{k_{\rm F}e^2}{q2\pi\hbar}\delta\frac{1}{2(qR)^2} \times \left[\frac{7}{4}\frac{1}{\sqrt{(1-\delta^2)^5}} - \frac{1}{\sqrt{(1-\delta^2)^3}}\right] \tag{1.76}$$

Using these results for the conductivities, we can find an expression for the density-density response function 1.41. For $\delta \ll 1$ and $qR \gg 1$,

$$\Pi_{00} = \frac{q^3}{q^3\left(\frac{d\eta}{d\mu}\right)^{-1} - 8\pi i\hbar\omega k_F\left(1 + (2k_F R)^{-1} + \frac{3}{8}(qR)^{-2}\right)}, \tag{1.77}$$

where $\frac{dn}{d\mu}$ is the compressibility of the $\nu = \frac{1}{2}$ state, which is defined as

$$\frac{dn}{d\mu} \equiv \Pi_{00}(\mathbf{q} \to 0, \omega \to 0) = \frac{3m^*}{8\pi\hbar^2} \tag{1.78}$$

We will use this expression for $\Pi_{00}$ to calculate $\rho_D$ in the next section.

# Chapter 2

# Attempts to calculate $\rho_D$

## 2.1 The approach of Ussishkin and Stern

We follow the approach of Ussishkin and Stern[1] to calculate the Coulomb drag for $\nu$ slightly away from 1/2. Ussishkin and Stern calculate $\rho_D$ for $\nu = 1/2$, i.e. zero effective magnetic field for the composite fermions, and we attempt to perturb this calculation with a small magnetic field $\Delta B$. Note that we consider the case of two identical layers ($\Pi_{(1)} = \Pi_{(2)} = \Pi$).

The integrand for $\rho_D$ involves (the square of) the expression $|U_{sc}(q,\omega)|\mathrm{Im}\Pi$. As in equation (12) of [1], we may re-write this as follows:

$$|U_{sc}(q,\omega)|\mathrm{Im}\Pi = -\mathrm{Im}(\Pi^{-1}) \left| \frac{U_b}{(\Pi^{-1}+V_b+U_b)(\Pi^{-1}+V_b-U_b)} \right|, \tag{2.1}$$

where $V_b(\mathbf{q}) = \frac{2\pi e^2}{\epsilon q}$ and $U_b(\mathbf{q}) = \frac{2\pi e^2}{\epsilon q} e^{-qd}$ are Fourier components of the bare Coulomb potentials for intralayer and interlayer interactions, respectively, and $\epsilon$ is the dielectric constant. Our first simplification is to make the following approximations:

$$\begin{aligned} U_b + V_b &\simeq 2V_b \\ V_b - U_b &\simeq qdV_b. \end{aligned} \tag{2.2}$$

To obtain these approximations for $qd$ small we expand the exponential $e^{-qd} \simeq 1 - qd$

and then take highest order terms, and for the the $U_b$ term in the numerator, we keep only the leading term [i.e. $V_b$].

Let $\beta = (d\eta/d\mu)^{-1}$. A simple computation shows that

$$\mathrm{Im}\Pi^{-1} = -8\pi\hbar\omega k_F q^{-3}\left(1 + (2k_F R)^{-1} + \frac{3}{8}(qR)^{-2}\right) \tag{2.3}$$

Applying all our approximations so far, we find that $|\Pi^{-1} + V_b + U_b)|^2$ equals

$$\left(\beta + \frac{4\pi e^2}{q\epsilon}\right)^2 + \left(\frac{8\hbar k_F \pi\omega}{q^3} + \frac{4\hbar\pi\omega}{q^3 R} + \frac{3\hbar k_F \pi\omega}{q^5 R^2}\right)^2. \tag{2.4}$$

and $|(\Pi^{-1} + V_b - U_b)|^2$ equals

$$\left(\beta + \frac{2d\pi e^2}{\epsilon}\right)^2 + \left(\frac{8\hbar k_F \pi\omega}{q^3} + \frac{4\hbar\pi\omega}{q^3 R} + \frac{3\hbar k_F \pi\omega}{q^5 R^2}\right)^2. \tag{2.5}$$

As in [1], $T_0$ is defined to satisfy the following equations:

$$T_0 = \frac{\pi e^2 nd}{\epsilon}(1+\alpha), \qquad \beta = \alpha \cdot \frac{2\pi e^2 d}{\epsilon}, \qquad 2T_0 = \frac{2\pi e^2 nd}{\epsilon} + n\beta. \tag{2.6}$$

Thus

$$\beta + \frac{2\pi de^2}{\epsilon} = 2T_0/n. \tag{2.7}$$

In the unperturbed calculation of [1], one made the approximations:

$$|(\Pi^{-1} + V_b + U_b)|^2 \simeq \left(\frac{4\pi e^2}{q\epsilon}\right)^2,$$

$$|(\Pi^{-1} + V_b - U_b)|^2 \simeq \left(\beta + \frac{2\pi de^2}{\epsilon}\right)^2 + \left(\frac{8\hbar k_F \pi\omega}{q^3}\right)^2. \tag{2.8}$$

To make these approximations one needs to compare various terms and decide which terms can be omitted. To do this we use the fact that $q \sim k_F(T/T_0)^{1/3}$ and $\omega \sim T$ is the region where most of the weight of the integral takes place.[1]

Let us keep the first approximation, and continue to keep all the terms of the second expression. Thus

$$|(\Pi^{-1}+V_b-U_b)|^2 \simeq \left(\beta+\frac{2\pi de^2}{\epsilon}\right)^2+\eta(q)^2\left(\frac{8\hbar k_F\pi\omega}{q^3}\right)^2. \tag{2.9}$$

Where $\eta(q)=1+(2k_FR)^{-1}+(qR)^{-2}$.

Recall that $d\mathbf{q}=2\pi q dq$. Thus the first integral we need to consider is:

$$\int_0^\infty \frac{64\pi^2\hbar^2\omega^2k_F^2q^{-6}\eta(q)^2\cdot(2\pi e^2/q\epsilon)^2}{(4\pi e^2/q\epsilon)^2\cdot(4T_0^2/n^2+(8\hbar k_F\pi\omega/q^3)^2\eta(q)^2)}\frac{2\pi q^3dq}{(2\pi)^2} \tag{2.10}$$

We can extract the constants to get

$$2\pi n^2\hbar^2\omega^2k_F^2\int_0^\infty \frac{q^3\eta(q)^2dq}{q^6T_0^2+(4n\hbar k_F\pi\omega)^2\eta(q)^2}. \tag{2.11}$$

Let $\eta=1+(2k_FR)^{-1}$. Then $\eta(q)=\eta+(qR)^{-2}$. Let us therefore expand the above integrand in $\eta(q)$ around $\eta$. Suppressing the constant factor $2\pi n^2\hbar^2k_F^2\omega^2$ temporarily, we find that the integral equals

$$\int_0^\infty \frac{q^3\eta^2dq}{q^6T_0^2+(4n\hbar k_F\pi\omega)^2\eta^2}$$

$$+\int_0^\infty \frac{1}{q^2R^2}\left(\frac{2\eta q^3}{q^6T_0^2+(4n\hbar k_F\pi\omega)^2\eta^2}-\frac{2(4n\hbar k_F\pi)^2\omega^2\eta^3q^3}{(q^6T_0^2+(4n\hbar k_F\pi\omega)^2\eta^2)^2}\right)dq.$$

$$=\int_0^\infty \frac{q^3\eta^2dq}{q^6T_0^2+(4n\hbar k_F\pi)^2\eta^2}+\frac{1}{R^2}\int_0^\infty\frac{2\eta q^7T_0^2\omega^2dq}{(q^6T_0^2+(4n\hbar k_F\pi\omega)^2\eta^2)^2}. \tag{2.12}$$

Now make the substitution $q=(4n\hbar k_F\pi\omega\eta/T_0)^{1/3}z$. Then the integral becomes:

$$\int_0^\infty \frac{(4n\hbar k_F\pi\eta\omega/T_0)z^3\eta^2(4n\hbar k_F\pi\eta\omega/T_0)^{1/3}dz}{(z^6+1)(4n\hbar k_F\pi\omega)^2\eta^2}$$

$$+\frac{1}{R^2}\int_0^\infty\frac{2\eta z^7(4n\hbar k_F\pi\eta\omega/T_0)^{7/3}T_0^2(4n\hbar k_F\pi\eta\omega/T_0)^{1/3}dz}{(z^6+1)^2(4n\hbar k_F\pi\omega)^4\eta^4)}. \tag{2.13}$$

Simplifying, and adding back in the constant $2\pi n^2\hbar^2k_F^2\omega^2$, this becomes:

$$\frac{\eta^{4/3}}{8\pi}\left(\frac{4n\hbar k_F\pi\omega}{T_0}\right)^{4/3}\int_0^\infty\frac{z^3dz}{z^6+1}+\frac{\eta^{-1/3}}{\pi R^2}\left(\frac{4n\hbar k_F\omega}{T_0}\right)^{2/3}\int_0^\infty\frac{z^7dz}{(z^6+1)^2}. \tag{2.14}$$

Evaluating the integrals, this equals:

$$\frac{\eta^{4/3}}{24\sqrt{3}}\left(\frac{4n\hbar k_F\pi\omega}{T_0}\right)^{4/3}+\frac{2\eta^{-1/3}}{18\sqrt{3}R^2}\left(\frac{4n\hbar k_F\omega}{T_0}\right)^{2/3}. \tag{2.15}$$

We must consider

$$\frac{1}{8\pi^2}\frac{h}{e^2}\frac{1}{Tn^2}\int_0^\infty\frac{1}{\sinh^2(\omega\hbar/2T)} \tag{2.16}$$

applied to this expression. Make the change of variables $\omega = 2T/\hbar \cdot y$.

Up to the constant $h/(8\pi^2n^2e^2)$ this is

$$\frac{\eta^{4/3}}{12\sqrt{3}}\left(\frac{8n\pi k_FT}{T_0}\right)^{4/3}\int_0^\infty\frac{y^{4/3}}{\sinh^2 y}+\frac{2\eta^{-1/3}}{9\sqrt{3}R^2}\left(\frac{8n\pi k_FT}{T_0}\right)^{2/3}\int_0^\infty\frac{y^{2/3}}{\sinh^2 y}. \tag{2.17}$$

The first term (with $\eta = 1$) is exactly the integral that occurs in [1]. Thus the second integral is the *new* term we are seeking due to the presence of the additional field. Unfortunately, however, this second integral is divergent. The essential reason for this is that all our approximations are valid only for $qR \gg 0$. This applies to the expansion of $\eta(q)$ around $\eta$ but also to the initial derivation of equation 1.77. One potential remedy to this approach is to work with our approximation to $\Pi$ only in some fixed region (say $qR \gg 0$) and introduce a cut-off point in the integral. One problem with this approach is that breaking up the $q$-integral in this way would prevent us from a closed form evaluation of these integrals. Another more significant issue is choosing where to make the exact value of the cut, since the answer may be sensitive to this choice. We will however try and regularize the divergence in the second integral by this method. Since $\left(\frac{\hbar\omega}{k_BT_0}\right)^{1/3} \sim q/k_F$ one may regard the small $Rq$ domain as being the small $\omega$ domain. Then we may work with our new approximation to $\Pi$ in the large $\omega$ domain $\omega > \epsilon$ and cut off the divergent integral otherwise. This will enable us to utilize our exact calculations above, but we will see there is some issue with the choice of $\epsilon$. Since the first integral converges we leave it as is. We replace the second integral with

$$\int_\epsilon^\infty\frac{y^{2/3}}{\sinh^2 y}.$$

In order to suppress any dependence on some parameter $\epsilon$, we assume that $\epsilon$ is not too close to zero. Note that the integral

$$\int_\epsilon^\infty \frac{1}{y^{4/3}\cosh^2 y}$$

is very small if $\epsilon$ is not too small, since cosh grows very quickly. Thus we may replace the integral above by

$$\int_\epsilon^\infty \frac{y^{2/3}}{\sinh^2 y} - \frac{1}{y^{4/3}\cosh^2 y}.$$

On the other hand, this integral does converge when $\epsilon \to 0$, and since the integrand is small at zero we may also estimate

$$\int_\epsilon^\infty \frac{y^{2/3}}{\sinh^2 y} \simeq \int_0^\infty \frac{y^{2/3}}{\sinh^2 y} - \frac{1}{y^{4/3}\cosh^2 y}.$$

The form of the latter integral is chosen to have these properties:

1. The integrand decreases exponentially quickly as $y$ gets larger,
2. The integrand function $f(y)$ satisfies $f(y) - y^{-4/3} \to 0$ as $y \to 0$.

However, this is still an arbitrary choice as we could have replaced $\cosh^2 y$ by $\cosh^3 y$, for example. One issue to be worried about is what $\epsilon$ really is. The $(qR)^{-2}$ perturbation makes its main contribution to the integral when $y > 1$, or when $\omega > 2T/\hbar$.

Returning to our integral, we note that $k_F = \sqrt{4\pi n}$, so $n = k_F^2/4\pi$. Thus

$$\frac{h}{8\pi^2 n^2 e^2} = \frac{2h}{e^2 k_F^4} \tag{2.18}$$

and the integral becomes

$$\frac{2h}{e^2 k_F^4}\left(\frac{\eta^{4/3}}{12\sqrt{3}}\left(\frac{2k_F^2 k_F T}{T_0}\right)^{4/3}\int_0^\infty \frac{y^{4/3}}{\sinh^2 y}\right.$$

$$\left.+\frac{2\eta^{-1/3}}{9\sqrt{3}R^2}\left(\frac{2k_F^3 T}{T_0}\right)^{2/3}\int_0^\infty \frac{y^{2/3}}{\sinh^2 y} - \frac{1}{y^{4/3}\cosh^2 y}\right).$$

$$= \frac{h}{e^2}\frac{\eta^{4/3}}{6\sqrt{3}}\left(\frac{2T}{T_0}\right)^{4/3}\int_0^\infty \frac{y^{4/3}}{\sinh^2 y} + \frac{2h\eta^{-1/3}}{9\sqrt{3}R^2k_F^2}\left(\frac{2T}{T_0}\right)^{2/3}\int_0^\infty \frac{y^{2/3}}{\sinh^2 y} - \frac{1}{y^{4/3}\cosh^2 y}. \tag{2.19}$$

Recall that

$$\int_0^\infty \frac{y^{s-1}}{\sinh^2 y} = \frac{2}{2^{s-1}}\Gamma(s)\zeta(s-1).$$

Then our integral equals

$$= \frac{h}{e^2}\frac{\Gamma(\frac{7}{3})\zeta(\frac{4}{3})\eta^{4/3}}{3\sqrt{3}}\left(\frac{T}{T_0}\right)^{4/3} + \frac{h}{e}\left(\frac{T}{T_0}\right)^{2/3}\frac{1}{(k_F R)^2}\frac{\eta^{-1/3}2^{2/3}}{18\sqrt{3}}\int_0^\infty \frac{y^{2/3}}{\sinh^2 y} - \frac{1}{y^{4/3}\cosh^2 y}. \tag{2.20}$$

Note that since

$$\eta^{4/3} - 1 = \frac{2}{3k_F R}\left(1 + \frac{1}{12k_F R}\right)$$

we may approximate (letting $\eta^{-1/3} \simeq 1$) $\rho_D$ as

$$\frac{2}{3k_F R}\left(1 + \frac{1}{12k_F R}\right)\rho_{D_0} + 4a\frac{h}{e}\left(\frac{2T}{T_0}\right)^{2/3}\frac{1}{(k_F R)^2} \tag{2.21}$$

with

$$a \simeq \frac{1}{18\sqrt{3}}\int_0^\infty \frac{y^{2/3}}{\sinh^2 y} - \frac{1}{y^{4/3}\cosh^2 y}.$$

### 2.1.1 Analysis

The approach of this section seems to mirror and at least reproduce the results of Ussishkin-Stern. The main problem is the introduction of a "cut-off" $\epsilon$ at which to cut off the divergent integral. Even though our final answer does not directly depend on $\epsilon$, the choices we have made in regularizing our integral are in the end arbitrary and thus not mathematically rigorous. In particular, different choices could lead to arbitrarily different values of the constant $a$. All of these issues relate to the approximation of $\Pi$ given by equation 1.77. If this approximation is not sufficiently well behaved for $qR \ll 1$ it will always cause problems in computing $\rho_D$. Thus we are led to try and find a more sensitive approximation to $\Pi$ for small $qR$. Although this leads to the analysis of the next few sections, this is the only approach which yields a closed form solution that we can compute.

## 2.2 Higher Level Approximations

In the previous section, we saw that our expression for $\Pi$ (equation 1.77) based on the conductivities we approximated led to divergent integrals due to the breakdown of our approximation for small $qR$. Thus we return to the exact solutions for the conductivities and compare them with our previous computations. We recover our approximations for $\sigma_{xx}$, but return different results for the other two expressions.

### 2.2.1 Preliminaries: Bessel Functions, Part I

The exact conductivities derived in Chapter 1 by solving the linearized Boltzmann equation were summed in equations (B15), (B16), and (B18) of [8] in the limit $\tau \longrightarrow \infty$. For example, we have (B15):

$$\sigma_{xx} = \frac{ipe^2}{\pi\hbar}\frac{2r}{X^2}\left[-\frac{1}{2} + \frac{\pi r}{2\sin(\pi r)}J_r(X)J_{-r}(X)\right], \tag{2.22}$$

where $r \equiv \omega/\omega_c^*$, $X \equiv 2qp/k_F = qR$, $\omega_c^* = \frac{e(\Delta B)}{m^* c}$, and $p = \frac{2\pi n\hbar c}{e(\Delta B)}$. When the parameter $X$ is sufficiently small, one can expand with respect to $X$ to obtain an approximation of $\sigma$. The key identities one requires are

$$\Gamma(z+1) = z\Gamma(z), \qquad \Gamma(z)\Gamma(1-z) = \pi\csc(\pi z). \tag{2.23}$$

One also needs to know the expansion of the Bessel function. By definition, one has

$$J_M(X) = \frac{X^M}{2^M\Gamma(M+1)}\left(1 - \frac{X^2}{2(2M+2)} + \frac{X^4}{2\cdot 4(2M+2)(2M+4)} + O\left(X^6\right)\right). \tag{2.24}$$

These three facts together imply that

$$J_r(X)J_{-r}(X) = \frac{\sin(\pi r)}{\pi r}\left(1 + \frac{X^2}{2(r^2-1)} + \frac{3X^4}{8(r^2-4)(r^2-1)} + O\left(X^6\right)\right). \tag{2.25}$$

Substituting this into the defining formula for $\sigma_{xx}$ then gives

$$\sigma_{xx} = \frac{ipe^2}{2\pi\hbar}\left(\frac{r}{r^2-1} + \frac{3rX^2}{4(r^2-1)(r^2-4)} + O\left(X^4\right)\right). \tag{2.26}$$

When $X$ tends to zero we see that the leading term is

$$\sigma_{xx} \approx \frac{ipe^2}{2\pi\hbar}\cdot\frac{r}{r^2-1}. \tag{2.27}$$

and thus some algebra shows that

$$\sigma_{xx}(\omega^2 - (\omega_c^*)^2) \approx \frac{-ipe^2\omega\omega_c^*}{2\pi\hbar} = \frac{ipe^2\omega}{2\pi\hbar}\cdot\frac{e(\Delta B)}{m^*c} = \frac{e^2}{m^*}\frac{ip\omega e(\Delta B)}{2\pi\hbar c}$$

$$= \frac{e^2}{m^*}\cdot\frac{ie\cdot\omega(\Delta B)}{2\pi\hbar c}\cdot\frac{2\pi n\hbar c}{e(\Delta B)} = \frac{e^2 ni\omega}{m^*},$$

and thus

$$\sigma_{xx} \approx \frac{e^2 n}{m^*}\frac{-i\omega}{(\omega_c^*)^2 - \omega^2} + O\left(X^2\right), \tag{2.28}$$

which is equation (B20) of [8]. The other terms in (B20) can be derived in a similar manner. We call this the approximation to $\sigma_{xx}$ in the small $X$ and thus small $q$ regime, and denote this approximation by $\sigma_{xx,\text{small}}$.

### 2.2.2 $\sigma_{xx}$ approximations

The graphs in this section are drawn with respect to $X$, which is a dimensionless variable.

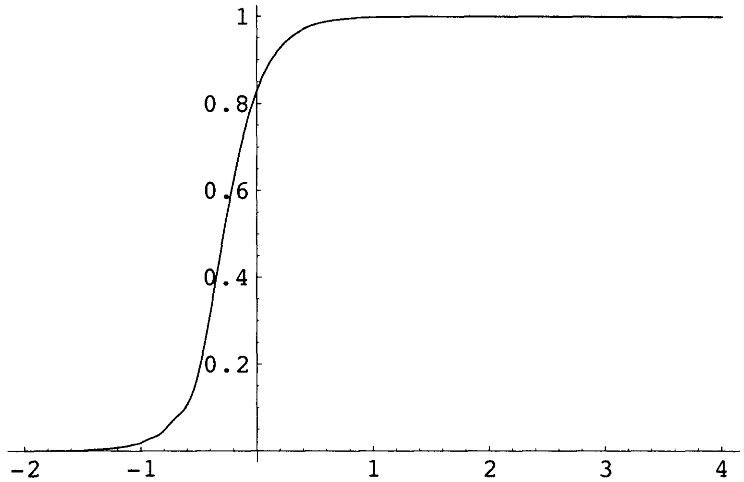


Figure 2-1: $y = \sigma_{xx}/\sigma_{xx,\text{small}}$ evaluated at $X = qR = 10^{-x}$, $r \sim 1$

This graph gives the ratio $\sigma_{xx}/\sigma_{xx,\text{small}}$ in the region $-\log_{10}(X) \in [-2, 4]$. Thus for $X = qR \ll 1$ we see that $\sigma_{xx,\text{small}}$ is a very good approximation to $\sigma_{xx}$. This gives a numerical confirmation of equation (B20) of [8]. Note that for this graph $r$ was chosen to be approximately 1, but the approximation continues to hold for other values of the Bessel function parameter $r = \omega/\omega_c^*$.

On the other hand, $\sigma_{xx}$ derived at the end of Chapter 1 gives the aproximation

$$\sigma_{xx} = -\frac{m^*}{2\pi\hbar^2}\frac{i\omega}{q^2}e^2\left(1 - \frac{1}{8(qR)^2} + O(\delta)\right) \tag{2.29}$$

This is supposed to be valid for $\delta$ small. Since $\delta = \omega/qv_F$, this is the large $q$-regime. We thus call this approximation $\sigma_{xx,\text{large}}$.

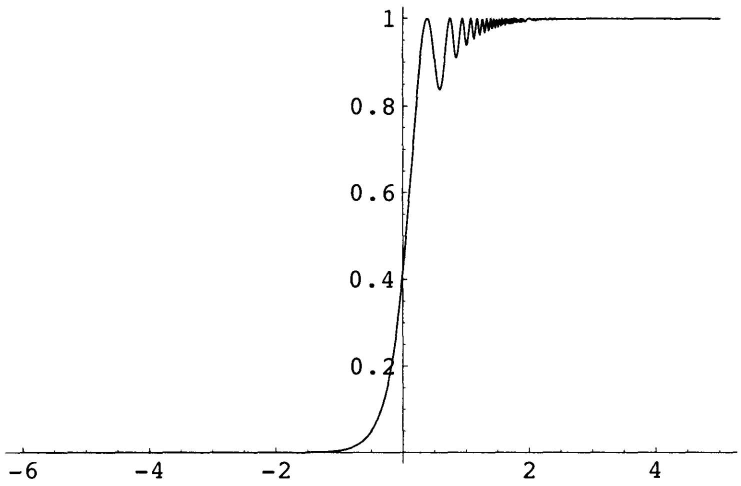


Figure 2-2: $y = \sigma_{xx}/\sigma_{xx,\text{large}}$ evaluated at $X = qR = 10^x$, $r \sim 1$

This graph gives the ratio $\sigma_{xx}/\sigma_{xx,\text{large}}$ in the large $qR$ regime, namely in the region $\log_{10}(X) \in [-6, 5]$. We see when $qR \gg 1$, $\sigma_{xx,\text{large}}$ gives a very good approximation to $\sigma_{xx}$.

### 2.2.3 $\sigma_{yy}$ approximations

We may perform the above approximations with $\sigma_{yy}$ instead of $\sigma_{xx}$. We note that

$$\sigma_{yy,\text{small}} = \sigma_{xx,\text{small}}.$$

This follows from the fact that the diagonal enteries in equation (B20) of [8] are equal. As in figure 1, we may compare $\sigma_{yy}$ to $\sigma_{yy,\text{small}}$.

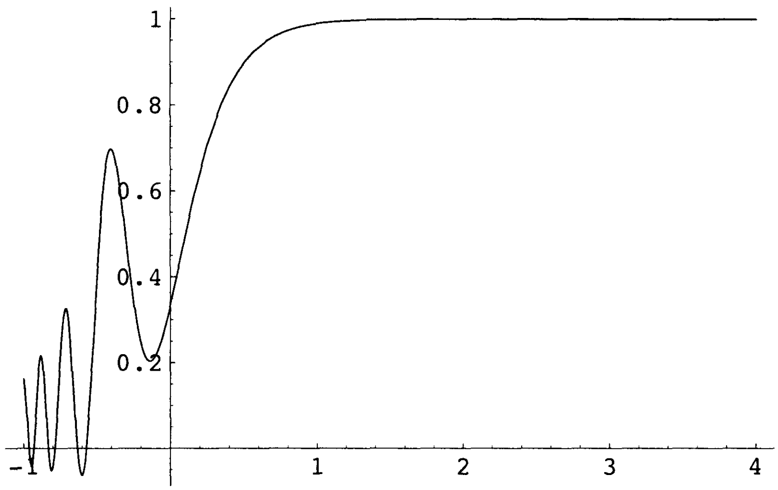


Figure 2-3: $y = \sigma_{yy}/\sigma_{yy,\text{small}}$ evaluated at $X = qr = 10^{-x}$, $r \sim 1$

This graph, logarithmic in the region $\log_{10}(X) \in [-1, 4]$ shows that for $qR \ll 1$, the approximation $\sigma_{yy} \approx \sigma_{yy,\text{small}}$ is good. In the large $qR$-regime, we have the approximation derived at the end of Chapter 1, which is

$$\sigma_{yy} \approx \frac{m^*}{2\pi\hbar^2}\frac{v_F e^2}{q}\left(1 + \frac{3}{8(qR)^2} + O(\delta)\right), \tag{2.30}$$

If we graph $\sigma_{yy}/\sigma_{yy,\text{large}}$, however, we get the following:

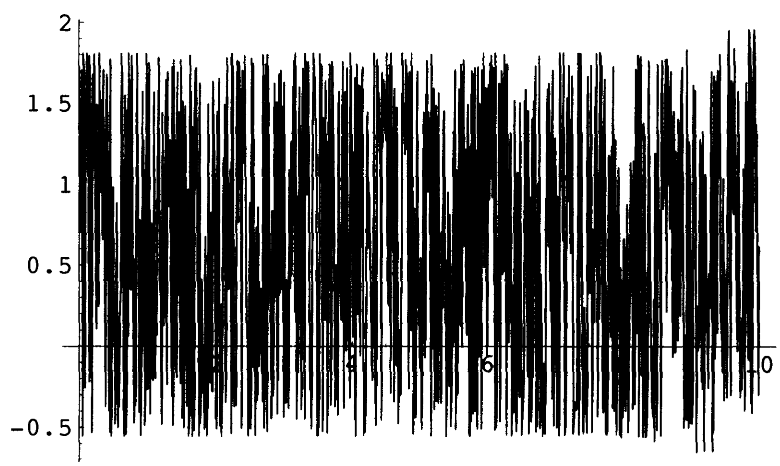


Figure 2-4: $y = \sigma_{yy}/\sigma_{yy,\text{large}}$ evaluated at $X = qR = 10^x$, $x \in [0, 10]$, $r \sim 1$

Even though the magnitude is roughly correct (it appears symmetric around 1) there are wild oscillations, even across the $x$-axis. The expression $\sigma_{yy,\text{large}}$ is always positive and monotonically decreasing: as a function of $q$ it is essentially $c/q$. The Bessel function terms in $\sigma_{yy}$, however, are wildly oscillating. This phenomenon did not arise in $\sigma_{xx}$. The reason is that in the large $q$-regime, the dominating term in the exact formula for $\sigma_{xx}$ (equation 2.22) comes from the constant $-1/2$, not from the Bessel functions! One possibility is that the results derived are right *on average*, (i.e. by smoothing out the oscillations, as seems to be correct) but do not actually give a good approximation to $\sigma_{yy}$? We follow up on this in the next section. Note that a similar phenomenon happens when comparing $\sigma_{xy}$ and $\sigma_{xy,\text{large}}$.

### 2.2.4 Resolving the approximations $\sigma_{yy}$ and $\sigma_{yy,\text{large}}$

Note that the approximations $\sigma_{yy}$ and $\sigma_{yy,\text{large}}$ in the last section were different but similar *on average*. To push this analysis further, we replace the value of $r$ by $r + i\epsilon$, for various real constants $\epsilon$, and take absolute values. For a particular value of $r$ we draw the graphs for $r + i$ and $r + 3i$, and we find the following:

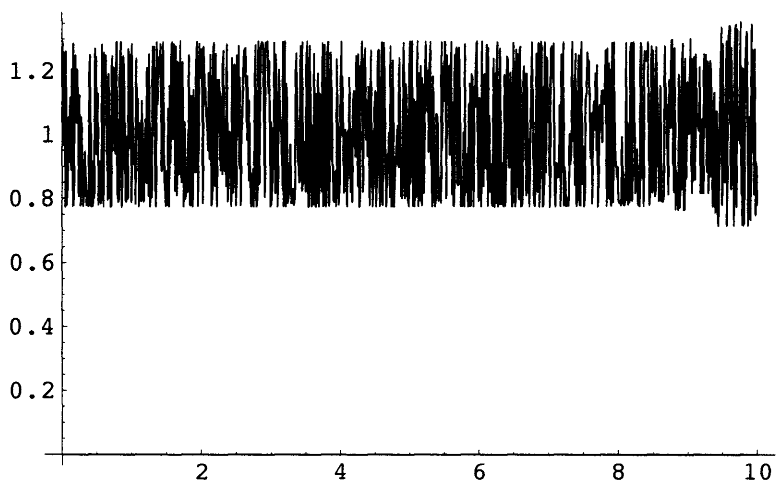


Figure 2-5: $y = ||\sigma_{yy}/\sigma_{yy,\text{large}}||$ evaluated at $X = qR = 10^x$, $x \in [0, 10]$, $r \sim 1 + i$

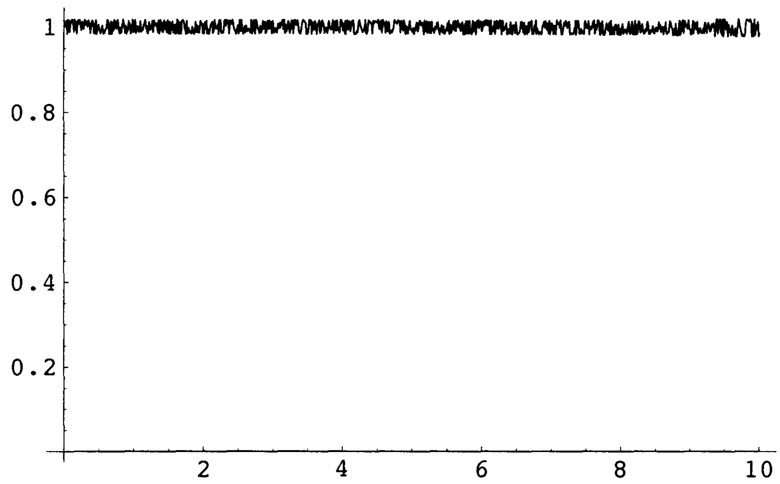


Figure 2-6: $y = \|\sigma_{yy}/\sigma_{yy,\text{large}}\|$ evaluated at $X = qR = 10^x$, $x \in [0, 10]$, $r \sim 1 + 3i$

Thus it appears that our approximation $\sigma_{yy,\text{large}}$ is accurate only after replacing $r$ by $r + i\epsilon$ and taking $\epsilon > 1$. We note that $\epsilon$ is analogous to the scattering term $1/\tau$ that appears in our exact results for the conductivities derived in Chapter 1. Thus, we can view the addition of the imaginary correction as the addition of scattering. This implies that the straightforward expansion of the Boltzmann equation (obtained in equation 2.30) ends up giving the wrong result; it essentially produces a dampened version of $\sigma_{yy}$ rather than the oscillating behavior. On the other hand, it does appear to give the correct answer after dampening all oscillating terms.

### 2.2.5 Analysis

In the first section we found that our approximations broke down at $q = 0$ and introduced divergences into the relevant integrals. This required choosing a "cut-off" at which to switch between our new approximation and the approximation of Ussishkin-Stern. In order to gain more insight into the nature of this cut-off, we studied in this section some of the local terms more explicitly using their description in terms of Bessel functions. However, it turned out that these explicit expressions varied considerably from the approximations in section one, and thus we could not infer anything about our previous analysis. On closer inspection, the approximation equation 2.30 obtained by expansion of the Boltzmann equation only produces the correct answer *on average*, and does not capture the analytic nature of $\sigma_{yy}$, but rather a damped variant where $r$ is replaced by $r + i\epsilon$ and $\epsilon > 1$. It is not clear to what extent this affects the analysis of the first section since the dampened form of $\sigma_{yy}$ may still be sufficient to compute $\rho_D$. One possibility is to work from the beginning using only the exact formulas of Simon-Halperin, which is our approach in the next section.

# Chapter 3

# Theoretical Approaches

## 3.1 Theoretical Approaches

If we use exact formulas in our integral for all relevant terms, then we (see below, equation 3.18) find that the expression for $\Pi$ is totally real. Initially this seems to be a serious problem as our integral calls for taking the imaginary part of $\Pi$. To account for this we use the standard technique of instead considering the retardation $\Pi_{\text{ret}}$ of $\Pi$, where a real parameter $x$ is replaced by $x + i\epsilon$ for some infinitesimally small $\epsilon$. This has the effect of introducting $\delta$ functions, as we now explain. The function $1/x$ is totally real for real values of $x$. Consider the retardation $(1/x)_{\text{ret}}$. We find that

$$\left(\frac{1}{x}\right)_{\text{ret}} = \frac{1}{x + i\epsilon} = (x - i\epsilon)/(x^2 + \epsilon^2). \tag{3.1}$$

Thus taking imaginary parts we find that

$$\text{Im}\left(\frac{1}{x}\right)_{\text{ret}} = \frac{-\epsilon}{x^2 + \epsilon^2}. \tag{3.2}$$

The expression $-\epsilon/(x^2+\epsilon^2)$ converges to zero as $\epsilon \to 0$ except at $x = 0$. This suggests that the limit may be a $\delta$ function concentrated at zero. To determine the scaling of this function we compute that

$$\int_{-\infty}^{\infty} \frac{\epsilon dx}{x^2+\epsilon^2} = \pi. \tag{3.3}$$

Thus the retarded version of Im $1/x$ is equal to $-\delta(x)/\pi$. We write this in the form

$$\mathrm{Im}\left(\frac{1}{x}\right)_{\mathrm{ret}} = -\frac{1}{\pi}\cdot\delta(x). \tag{3.4}$$

Since we will encounter functions with many poles, we will study integrals that consist of the sum of many delta functions integrated against some other functions. We try to develop a framework for studying such integrals. First, however, we recap some more facts about Bessel functions.

### 3.1.1 Preliminaries: Bessel Functions, Part II

We return again to formal analysis of Bessel functions in order to apply some exact formulas of Simon-Halperin.

Consider the following differential equation:

$$X^2\frac{d^2Y}{dX^2} + X\frac{dY}{dX} + (X^2 - n^2)Y = 0. \tag{3.5}$$

For generic positive values of $n$, there is a unique power series solution at $X = 0$. Let us posit a power series solution of the form

$$X^r \sum_{k=0}^{\infty} a_k X^k \tag{3.6}$$

for some rational number $r$ with $a_0 \neq 0$. The differential equation above leads (by equating coefficients) to many equations, the first of which is $a_0(r^2 - n^2) = 0$. When $r = n$, the subsequent equations become

$$a_1(2n+1) = 0$$

and

$$a_k(k^2+2nk)+a_{k-2}=0.$$

It follows that $a_k=0$ for $k$ odd and that

$$a_{2k}=\frac{-a_{k-2}}{4k^2+4nk}.$$

Determining $a_{2k}$ leads to the well known series expansion of the Bessel function (up to a constant). The traditional value of $a_0$ gives the following expression for $J_n(X)$:

$$J_n(X):=\frac{X^n}{2^n\Gamma(n+1)}\sum_{k=0}^{\infty}\frac{(-1)^k X^{2k}}{2^{2k}k!\Gamma(n+k+1)}. \tag{3.7}$$

Note that to deduce this expression, we use the fact that

$$\frac{\Gamma(n+k+1)k!}{\Gamma(n+k)(k-1)!}=k^2+nk=\frac{4k^2+4nk}{2^2}.$$

The Bessel functions $J_n(X)$ also has an asymptotic expansion for $X\gg 0$. This can be obtained by applying the method of stationary phase. The result is the approximation

$$J_n(X)\approx\sqrt{\frac{2}{\pi x}}\cdot\cos\left(x-\frac{2n+1}{4}\cdot\pi\right). \tag{3.8}$$

### 3.1.2 The calculations of Simon-Halperin

Recall the following from (B15), (B16) and (B18) of Simon-Halperin [8].

$$\sigma_{xx}=\frac{ipe^2}{\pi\hbar}\frac{2r}{X^2}\left(-\frac{1}{2}+\frac{\pi r}{2\sin(\pi r)}J_r(X)J_{-r}(X)\right)$$

$$\sigma_{xy}=\frac{pe^2}{\pi\hbar}\frac{\pi r}{2X\sin(\pi r)}\left[J_r(X)J_{-r}(X)\right]'.$$

$$\sigma_{yy}=\sigma_{xx}+\frac{ipe^2}{\pi\hbar}\frac{\pi}{\sin(\pi r)}J_{1+r}(X)J_{1-r}(X).$$

Let us estimate these expressions for $X\gg 0$ by using the asymptotic formula 3.8.

One finds that

$$\sigma_{xx} \sim -\frac{ipe^2 r}{\pi\hbar X^2} \tag{3.9}$$

$$\sigma_{xy} \sim \frac{pe^2 r}{\pi\hbar X^2} \cdot \frac{\cos(2X)}{\sin(\pi r)} \tag{3.10}$$

$$\sigma_{yy} \sim \frac{ipe^2}{\pi\hbar X} \cdot \frac{\cos(\pi r) - \sin(2X)}{\sin(\pi r)} \tag{3.11}$$

To derive these equations, one uses the sums to products identity for $\cos(a)\cos(b)$ and equation 3.8.

### 3.1.3 A new expression for $\Pi$

In this section we use our approximations for $\sigma_{xx}$, $\sigma_{xy}$ and $\sigma_{yy}$ to derive a theoretical expression for $\Pi$ using Bessel functions. Recall $r = \omega/\omega_c^*$ and $X = qR$. We have

$$K_{01}(q,\omega) = \frac{q^2}{ie^2}\sigma_{xy} = -\frac{ip\omega}{\pi\hbar\omega_c^* R^2 q} \cdot \frac{\cos(2qR)}{\sin(\pi\omega/\omega_c^*)} \tag{3.12}$$

This expression does not involve any complex quantities and thus is real. We also have the following two expressions for the coefficients $K$ given in equation 1.43:

$$K_{11}(q,\omega) - K_{11}(q,0) = \frac{-\omega}{ie^2}\sigma_{yy} \tag{3.13}$$

$$K_{00}^{-1}(q,\omega) - K_{00}^{-1}(q,0) = \frac{\omega e^2}{iq^2}\sigma_{xx}^{-1} \tag{3.14}$$

We also impose the following equation:

$$K_{11}(q,0) = \frac{-q^2}{24\pi m^*} \tag{3.15}$$

This expression also does not involve any complex expressions and is thus real. The expression for $\sigma_{yy}$ is purely imaginary, and thus $i\sigma_{yy}$ is also real. It follows that $K_{11}(q,w)$ is real function. We set $K_{00}^{-1}(q,0) = 0$. Thus

$$K_{00}(q,\omega) = \frac{iq^2}{\omega e^2}\sigma_{xx} \tag{3.16}$$

is real. Let

$$\Delta = K_{00}K_{11} + K_{01}^2. \tag{3.17}$$

Since $K_{01}$ is purely imaginary, its square is real. Also, $K_{00}$ and $K_{11}$ are real, as we have noted. Thus $\Delta$ is real. Finally, let

$$\Pi = \frac{K_{00}}{1 - \frac{8i\pi\hbar}{q}K_{01} - \left(\frac{4\pi\hbar}{q}\right)^2 \Delta}, \tag{3.18}$$

The numerator is real, as is the denominator ($K_{01}$ is purely imaginary, so $iK_{01}$ is real). Thus $\Pi$ is real. Therefore, to be able to interpret $\mathrm{Im}(\Pi)$ we must form the retardation of $\Pi$ by examining is poles.

### 3.1.4 The retardation $\Pi_{\mathrm{ret}}$ and $\mathrm{Im}(\Pi_{\mathrm{ret}})$

Since $\Pi$ is totally real, the expression $\mathrm{Im}(\Pi)$ only makes sense if we interpret it to be $\mathrm{Im}(\Pi_{\mathrm{ret}})$, where $\Pi_{\mathrm{ret}}$ is the retardation of $\Pi$. Explicitly this means that we replace $\omega$ by $\omega + i\epsilon$ for some arbitrary small $\epsilon$. As we found in our brief calculation above, $\mathrm{Im}(1/z)_{\mathrm{ret}}$ turns out to be $-\delta(z)/\pi$. For a general function $f(z)$ which is real for $z \in \mathbf{R}$ but may have simple poles, we therefore can interpret $\mathrm{Im}(f(z))$ to be

$$\mathrm{Im}(f(z)_{\mathrm{ret}}) = -\frac{1}{\pi} \sum_{f(\gamma)=\infty} \delta(z-\gamma) \cdot \mathrm{residue}(f,\gamma). \tag{3.19}$$

Here the residue of $f$ at $\gamma$ is defined in the usual way as the expression:

$$\mathrm{residue}(f,\gamma) = \lim_{z\to\gamma}(z-\gamma)f(z) \tag{3.20}$$

If we wish to understand the retardation of some function, we must therefore understand the location of the poles, and the residue of the function at those poles. Answering these two questions will allow us to construct a meaningful expression. By "meaningful" here we mean an expression that it makes sense to integrate, and thus $\delta$-functions are acceptable. Explicitly, we wish to apply this procedure to the integrand arising in $\rho_D$. However, we begin with a more theoretical discussion of how to integrate sums of many $\delta$-functions at once.

### 3.1.5 Estimating Sums of $\delta$-functions

Although integrating a single $\delta$-function is easy, it may (and will) be the case that the function we encounter has many poles, not all of which can be determined explicitly. Thus it will be useful to consider a general technique for integrating sums of *delta*-functions integrated against other expressions. This general formalism is the subject of this section which we will apply to our particular example later.

We begin with an example. Suppose we consider the function

$$K(q) = \sum_{n=1}^{1000} \delta(q-n). \tag{3.21}$$

If $f(q)$ is any continuous function, then by the standard property of delta functions we see that

$$\int_0^\infty K(q)f(q)dq = \sum_{n=1}^{1000} f(n) \tag{3.22}$$

On the other hand, suppose that the function $f(q)$ is relatively flat (ie. has small derivative). Then we may approximate

$$\int_0^{1000} f(q)dq \sim \sum_{n=1}^{1000} f(n). \tag{3.23}$$

Together, we obtain the approximation

$$\int_0^\infty K(q)f(q)dq \sim \int_0^{1000} f(q)dq \tag{3.24}$$

In other words, as far as the integral is concerned, the function $K(q)$ behaves as if it is a step function from 0 to 1000, even though the function $K(q)$ itself does not behave in this way. The reason such an approximation may be useful is as follows. Our integral expression for $\rho_D$ involves the quantity $\mathrm{Im}(\Pi)$. We interpret this to mean $\mathrm{Im}(\Pi_{\mathrm{ret}})$, which is a sum involving many $\delta$-functions. Since we only care about the impact of these $\delta$-functions inside an integral, it may be possible to replace the complicated sums of $\delta$-functions by a simple continuous function like the step function encountered above. To prepare for this situation, we will generalize the example above. Consider a continuous auxiliary function $g(q)$, and define a distribution $K(q)$ by considering the following sum:

$$K(q) := \sum_{\gamma_n \in S} \delta(q - \gamma_n) g(\gamma_n) \tag{3.25}$$

where $\gamma_n$ ranges over a set of points $S$ contained in an interval $\Omega$. If $\Omega = (0, 1000]$, $S$ is the set of integers in this range, and $g(q)$ is the constant function, we recover the situation described above. In practice, the points $\gamma_n$ will be the poles of $\Pi$ and the function $g$ will estimate the residue of this function at its poles, and the range $\Omega$ will be the interval in which the poles occur. The idea is to approximate the operator $\int K(q)$ for "well behaved" test functions.

Since we have the standard relation

$$\int \delta(q - \gamma) f(q) dq = f(\gamma),$$

it follows that

$$\int f(q) K(q) dq = \sum f(\gamma_n) g(\gamma_n). \tag{3.26}$$

On the other hand, consider the integral

$$\int_\Omega f(q) g(q) dq. \tag{3.27}$$

Suppose we try to compute the integral of equation 3.27 by using a Riemann sum.

We can estimate the integral by

$$\frac{1}{C} \sum f(\gamma_n) g(\gamma_n), \tag{3.28}$$

where $C$ is a constant defined as the "density" of poles in $\Omega$, i.e. $\#S/\text{Area}(\Omega)$.

Comparing equation 3.26 and 3.28 we are led to the approximation:

$$\int f(q)K(q)dq = \frac{1}{C}\int_{\Omega} f(q)g(q)dq. \tag{3.29}$$

Therefore, despite that $K(q)$ is a sum of $\delta$ functions and $g(q)/C$ is continuous, we may replace $K(q)$ by $g(q)/C$, as long as we restrict ourselves to functions that occur in integrals. We write this as

$$K(q) \equiv g(q)/C \tag{3.30}$$

As an example, suppose that $\Omega = (0, M]$ for some large integer $M$, that $g(x) = 1$, and that $S$ is the set of integers in $\Omega$. Then $C = 1$, and our approximation 3.29 is

$$\int_0^M f(q)dq \simeq \sum_{n=1}^{M} f(n). \tag{3.31}$$

Of course this is a better approximation for some functions $f(n)$ than others. In general, we assume that the derivative of $g$ and $f$ are not so large.

Suppose now we consider the alternating sum of delta functions

$$K(q) := \sum_{\gamma_n \in S} \delta(q - \gamma_n)g(\gamma_n)(-1)^n. \tag{3.32}$$

To estimate the integral

$$\int K(q)f(q)dq$$

we must approximate the sum

$$\sum f(\gamma_{2n})g(\gamma_{2n}) - f(\gamma_{2n-1})g(\gamma_{2n-1}) \tag{3.33}$$

If we assume that the test function $f$ has a smaller derivative than $g$, we may approximate this by

$$\frac{1}{2}\sum f(\gamma_n)g'(\gamma_n)d\gamma_n \simeq \frac{1}{2}\int_{\Omega} f(q)g'(q)dq. \tag{3.34}$$

We write this for future reference as

$$\int K(q)f(q)dq = \frac{1}{2}\int_{\Omega} f(q)g'(q)dq \tag{3.35}$$

and as above we also write

$$K(q) \equiv \frac{1}{2}\cdot g'(q) \tag{3.36}$$

Note that the constant $C$ does not occur in this formula. As an example, let $\Omega = (0, M]$, let $\gamma_n = n/N$ for $0 < n < NM$, and let $g(x) = x$. Then we are approximating

$$\frac{1}{2}\int_0^M f(q)dq = \sum_1^{NM} f\left(\frac{n}{N}\right)\frac{(-1)^n n}{N}.$$

For example, if $f(q) = 1$, then the LHS is equal to $M/2$, while the right hand side is equal to

$$\frac{NM-1}{2N} = \frac{M}{2} - \frac{1}{2N},$$

which, for large $N$. is in agreement with $M/2$.

### 3.1.6 An Alternate Approach

We apply the techniques of the last section to approximate $\mathrm{Im}(\Pi_{\mathrm{ret}})$, as given by the formulas from equations 3.9, 3.10, 3.11 and equation 3.18.

The expression for $1/\Pi$ can be written as follows:

$$1/\Pi = -\frac{4\pi\hbar^2}{3m^*} + \frac{8\pi\hbar k_F\omega\cot(\pi r(\omega))}{q^3} + \frac{8\pi\hbar\omega\csc(\pi r(\omega))}{q^2}\cdot\cos(2Rq) -$$

$$\frac{8\pi m\omega^2\csc(\pi r(\omega))^2}{q^4}\cdot\cos(2Rq)^2 - \frac{8\pi\hbar k_F\omega\csc(\pi r(\omega))}{q^3}\cdot\sin(2Rq). \tag{3.37}$$

What are the dominating terms in our expression for $1/\Pi$? Including the constant there are five terms, which we label $T_1$, $T_2$, $T_3$, $T_4$ and $T_5$. All trigonometric terms have roughly equal order. Thus, we may estimate

$$T_3/T_2 \sim \frac{q}{k_F}, \qquad T_4/T_2 \sim \frac{m^*\omega}{\hbar k_F q} = \frac{\omega}{v_F q}, \qquad T_5/T_2 \sim 1 \tag{3.38}$$

Since $q \ll k_F$ and $\frac{\omega}{v_F q} \ll 1$, it follows that $T_2$ and $T_5$ are the dominating terms, and possibly $T_1$. Numerical calculations suggest that the main weight of the integral takes place in the region where $T_5, T_2 \gg T_1$. This is a "middle" region of $Rq$ where the Bessel function approximations (which are valid for large $Rq$) are still valid. In this range we may estimate

$$1/\Pi \sim \frac{8\pi\hbar k_F\omega \cot(\pi r(\omega))}{q^3} - \frac{8\pi\hbar k_F\omega \csc(\pi r(\omega))}{q^3} \cdot \sin(2Rq)\,. \tag{3.39}$$

This expression naturally simplifies to the following:

$$1/\Pi \sim \frac{8\pi\hbar k_F\omega \csc(\pi r(\omega))}{q^3}\left(\cos(\pi r(\omega)) - \sin(2Rq)\right). \tag{3.40}$$

The next step is to substitute this function into equation 2.1, substitute $\omega + i\epsilon$ for $\omega$, and attempt to determine the poles and residues of the resulting expression in order to re-write it as a sum of $\delta$-functions. The effect of these substitutions is a complicated expression, however, for which it is hard to determine the poles. Some numerical analysis and some simplifications suggest that the methods of the previous section may lead to an expression that can be integrated, but all our attempts so far have resulted in expressions too complicated to integrate (numerically or otherwise), so we omit them here.

### 3.1.7 Analysis

Although the approach of this section theoretically might work, practically it becomes too difficult to explicitly determine the location of all the poles and their residues. If an accurate description of the poles and residues could be determined, then our techniques of simplifying large sums of $\delta$-functions could be applied to the estimation of the integrand of $\rho_D$, and then hopefully to $\rho_D$ itself. However, attempts to use various formulas obtained in this way have not succeeded. One relevant issue is that it is not clear if the methods will apply rigorously to our particular example. A second issue is that the numerical integrals that arise are still too complicated to produce good data with Mathematica. One possible future direction is to test the accuracy of our approximations of the poles of various expressions that we encounter. Another approach is to work numerically with a fixed (but small) $\epsilon > 0$ rather than the retarded form with $\epsilon$ arbitrarily small, and perform numerical computations; this would have the effect of smoothing away the $\delta$-functions. Although we have not managed to successfully carry out these steps, they are worthy of further consideration.

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